

Mixmaster chaos

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Abstract

The significant discussion about the possible chaotic behavior of the mixmaster cosmological model due to Cornish and Levin [J.N. Cornish and J.J. Levin, Phys. Rev. Lett. 78 (1997) 998; Phys. Rev. D 55 (1997) 7489] is revisited. We improve their method by correcting nontrivial oversights that make their work inconclusive to precisely confirm their result: “The mixmaster universe is indeed chaotic”.

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The Bianchi IX (mixmaster) cosmological model was investigated by Belinskii, Khalatnikov and Lifshitz (BKL), clarifying the nature of cosmological singularities (see [1] and Refs. therein), and by Misner, in his attempt to solve the cosmological horizon problem [2]. The subsequent emergence of chaos in maps associated with this model attracted the attention of several other researchers in the following years. Later, it was realized that standard chaotic indicators used in previous works, like Lyapunov exponents, are not invariant under space-*time* diffeomorphisms (see [3] for a comprehensive discussion). Since then, a number of contributions looking for coordinate independent manifestations of chaos have appeared in the literature, and the mixmaster cosmology has become one of the main paradigms of deterministic chaos in General Relativity. Even though some important results have been obtained, a satisfactory comprehension of chaos in this system remains to be achieved (see [4] for a critical review). Among the most promising proposals we cite the efforts of Szydlowski and collaborators in treating the problem in terms of (invariant) curvature [5]. This approach, however, breaks down in the mixmaster case [4]. Another significant progress was recently made by Cornish and Levin (CL) [6]. Employing fractal techniques, the authors obtained evidences of chaotic transients in the mixmaster model. Their work is, however, inconclusive due to conceptual flaws. Accordingly, the existence of invariant chaos in the mixmaster cosmological model is still an open question.

In dynamical systems, chaos seems to be the rule rather than the exception (at least for generic autonomous systems with three or more equations). Based on this fact, there is small risk in declaring in advance that a system is chaotic. In addition, even in nonchaotic systems, technical errors in general lead to some kind of numerical chaos. Therefore, the relevance of stating that a particular system is chaotic stays on the precision of the proof and the nature of the chaos revealed by it [7]. In the case of mixmaster universe it stays also on the need of a proper meaning for invariant chaos in General Relativity.

The aim of this communication is to show unequivocally that the mixmaster dynamics is chaotic in a precise sense. First we review critically the work of CL, pointing out some key oversights. Second, we propose modifications and corrections of the CL's procedure. Next, this new procedure is employed to the approximated dynamics and then to the exact dynamics. From the full dynamics analysis we conclude that the mixmaster model does evolve chaotically.

The work of CL is based on enlarging the three outcomes of the mixmaster universe and in studying the invariant set through fractal methods of chaotic scattering. The existence of a fractal structure in the invariant set would imply chaotic transients. First they studied an approximation of the exact dynamics, the Farey map, whose invariant set could be exhibited explicitly. For this map, the topological entropy and the information dimension of the invariant set were computed. Then, by numerically computing the information dimension for a choice of the outcomes, the full dynamics was studied. Based on the noninteger value of the resulting information dimension, the authors declared that the system is chaotic.

In checking the CL's papers we see that: (1) Concerning the Farey map, the invariant set exhibited, a Cartesian product of periodic points of the epoch-era BKL map [1], is not invariant. In contrast to the authors' belief, only points with the same period in both Cartesian components are periodic points of the Farey map. Consequently, the conclusions obtained from this 'invariant set' are wrong or at best suspicious. For instance, their result for the topological entropy is twice as bigger as the correct value because what they found was the square of the number of periodic points that really exist. (2) In the full dynamics counterpart, the information dimension was computed from the box-counting dimension of the basin boundaries, based on a result presented in Ref. [8] - the information dimension of a set may be given by the box-counting dimension of an *optimal* fraction of the set (see [9], p. 83) - that does not apply to this case because the optimal condition is not satisfied. Moreover, the initial conditions, given near the moment of maximal expansion, were parametrized by parameters from Chernoff-Barrow (CB) map and the outcomes were defined in terms of one of these parameters [10]. Since the CB map cannot be applied far from the cosmological singularities, this parametrization is violated during the integration and is recovered only when the Kasner approximation becomes valid. Accordingly, the outcomes are not well defined all the time and they make sense only when a Kasner approximation is made. Consequently, the full dynamics analysis is not really full. (3) Both approximated and full analyses involve the idea of a *full repellor* as a set of the system itself. This concept is unclear because repellers are functions of the outcomes, and therefore are sets that are defined only after the definition of the outcomes. The origin of the problem seems to be in some confusion between nonattracting invariant sets (repellers) and the set of periodic points.

When considered the above oversights, the CL's work becomes inconclusive. In spite of that, the authors left a great contribution: The suggestion of using fractal invariant methods to investigate the mixmaster model¹. In this communication we will follow this idea as close as possible of the CL's work in order to allow a direct comparison by the reader. But different from CL, we will choose the box-counting dimension as the invariant indicator of chaos [9]. The topological entropy and the information dimension, employed by CL, are unsuitable indicators of chaos in the mixmaster system: The topological entropy is a function of the time parametrization because of its dependence on the period of the periodic points; the information dimension is difficult to be computed since it is difficult to verify the optimal condition mentioned above (we stress that CL were not able to obtain such dimension in the case of the full dynamics). The box-counting dimension, on the other hand, is diffeomorphism invariant and can be easily computed by applying the uncertainty exponent method [9].

The Einstein equations expressed in terms of the three spatial scales of the model $(g_{ij}) = \text{diag}(a^2, b^2, c^2)$ lead to the motion equations [1]

$$(\ln a^2)'' = (b^2 - c^2)^2 - a^4, \text{ et cyc.}, \quad (1)$$

together with the constraint equation

$$(\ln a^2)'(\ln b^2)' + 2a^2b^2 - a^4 + \text{cyc.} = 0. \quad (2)$$

A prime denotes $d/d\tau$, where $dt = abc d\tau$ and t is the cosmological time. It is convenient to introduce α_1 , α_2 and α_3 as $\alpha^{(a)} = \ln a$, $\alpha^{(b)} = \ln b$ and $\alpha^{(c)} = \ln c$ in increasing order of the time derivatives. That is, $\dot{\alpha}_1 \leq \dot{\alpha}_2 \leq \dot{\alpha}_3$, where a dot denotes $d/d\Omega$ and Ω is the time $\ln(abc)/3$. The dynamics near the cosmological singularity is approximated by the CB map [10], obtained from a parametrization of the surface of section $\alpha_\kappa = 0, \dot{\alpha}_\kappa > 0$ by

$$\begin{aligned} \alpha_1 &= 3\Omega q_1(u, v), & \alpha_2 &= 3\Omega q_2(u, v), & \alpha_3 &= 3\Omega q_3(u, v), \\ \dot{\alpha}_1 &= 3p_1(u), & \dot{\alpha}_2 &= 3p_2(u), & \dot{\alpha}_3 &= 3p_3(u), \end{aligned} \quad (3)$$

where $q_1 = 1/(1+v+uv)$, $q_2 = \delta_{\kappa 3}(v+uv)/(1+v+uv)$, $q_3 = \delta_{\kappa 2}(v+uv)/(1+v+uv)$, $p_1 = -u/(1+u+u^2)$, $p_2 = \delta_{\kappa 2}(1+u)/(1+u+u^2) + \delta_{\kappa 3}(u+u^2)/(1+u+u^2)$

¹Fractal techniques were also introduced by Demaret and De Rop [11] to study an approximation of Bianchi IX cosmology.

and $p_3 = \delta_{\kappa 2}(u + u^2)/(1 + u + u^2) + \delta_{\kappa 3}(1 + u)/(1 + u + u^2)$, for $0 \leq u, v < \infty$. The evolution of the parameters u and v is then determined by the Farey map,

$$F(u, v) = \begin{cases} (u - 1, v + 1) & \text{if } u \geq 1 \text{ (change of epoch)} \\ (u^{-1} - 1, (v + 1)^{-1}) & \text{if } u < 1 \text{ (change of era).} \end{cases} \quad (4)$$

The scale factors² $(\alpha_1, \alpha_2, \alpha_3) = (\alpha^{(\mu)}, \alpha^{(\nu)}, \alpha^{(\rho)})$ evolve as $\dot{\alpha}_1 \leq \dot{\alpha}_2 \leq \dot{\alpha}_3 \mapsto \dot{\tilde{\alpha}}_1 \leq \dot{\tilde{\alpha}}_2 \leq \dot{\tilde{\alpha}}_3$, where $(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3) = (\alpha^{(\nu)}, \alpha^{(\mu)}, \alpha^{(\rho)})$ and $\kappa = 2$ except in the transition to the last epoch of each era, where $(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3) = (\alpha^{(\nu)}, \alpha^{(\rho)}, \alpha^{(\mu)})$ and κ becomes 3. The Farey map can be equivalently defined as,

$$\bar{F}(\bar{u}, \bar{v}) = \begin{cases} (\bar{u} - 1, \bar{v} + 1) & \text{if } \bar{u} \geq 2 \text{ (change of epoch)} \\ ((\bar{u} - 1)^{-1}, \bar{v}^{-1} + 1) & \text{if } \bar{u} < 2 \text{ (change of era),} \end{cases} \quad (5)$$

for $1 \leq \bar{u}, \bar{v} < \infty$. Since $\bar{F} = f^{-1} \circ F \circ f$ (\bar{F} and F are conjugate), where $f(\bar{u}, \bar{v}) = (\bar{u} - 1, \bar{v} - 1)$, the parametrization corresponding to \bar{F} is not just (3) (as used by CL) but, (3) with q_i and p_i replaced by $\bar{q}_i = q_i \circ f$ and $\bar{p}_i = p_i \circ f$, for $i = 1, 2, 3$.

First let us consider the approximated dynamics in order to have some insight into the problem. In taking relative quantities in (3), the overall scale factor Ω is canceled, and we are led to a stationary phase space $u \times v$ where three outcomes can be identified: $(u, \alpha_3) = (\infty, \alpha^{(a)})$, $(u, \alpha_3) = (\infty, \alpha^{(b)})$ and $(u, \alpha_3) = (\infty, \alpha^{(c)})$. Since only rational values of u are led to these outcomes by forward iteration of the map, the corresponding invariant set contains almost every point of the phase space and it tells us nothing about chaos. However, some useful conclusions can be obtained if we enlarge the outcomes. It is made by defining the three outcomes for $u > u_{out} - 1$ instead of $u = \infty$, which transform the system in an exit (scattering) system (the original Farey exit map was defined by CL [6]). These outcomes are equivalent to $\dot{\alpha}_3/(\dot{\alpha}_1 + \dot{\alpha}_2 + \dot{\alpha}_3) > \hat{p}_3(u_{out})$, where $\hat{p}_3(u) = (u + u^2)/(1 + u + u^2)$. The dynamics of this exit system is determined by a nonattracting invariant set - the repellor [9]. The repellor is a zero measure set consisting of a countable number of unstable periodic orbits surrounded by an uncountable number of nonperiodic orbits. A fractional box-counting dimension for the

²The indices μ, ν, ρ are permutations of a, b, c .

repellor implies³ the exit system to be chaotic in the sense of having a chaotic transient: The system evolves chaotically for a period of time before being scattered. Concerning the original system, the system is typically never scattered (except for particular choices of the initial conditions) and hence it would evolve chaotically forever. Therefore, if the exit system is chaotic, the original system is also chaotic.

In Fig. 1 we show, in three different colors, the basins of initial conditions for the outcomes defined by $u_{out} = 7$ (the same used by CL [6]). It can be shown that magnifications of the basin boundary lead to figures with the same complex structure of Fig. 1, where the three colors (basins) are always present. The basin boundary is the future invariant set, and its box-counting dimension estimated from the uncertainty exponent method [9] results $D_0 = 1.87 \pm 0.01$. Since the repellor is the intersection of the future and the past invariant sets, it is equivalent to study any one of these sets. We prefer the future invariant set because of its easier geometric interpretation. Therefore, the above result for D_0 is enough to conclude that the original map is chaotic. In varying the outcomes, D_0 becomes an increasing function of u_{out} as showed in Fig. 2a. When the outcomes are made arbitrarily small, D_0 approaches 2. This behavior is completely different in a nonchaotic system, since in that case D_0 is integer and does not approach the phase space dimension when the outcomes are reduced.

Now we repeat the procedure for the full dynamics, numerically integrating the mixmaster equations. In order to allow comparisons with the approximated study, we take the initial condition defined by (3) for $1 < u, v < 2$ ($\kappa = 2$), and Ω the negative solution of the constraint equation. The integration is performed in the negative direction of Ω (approaching the singularity). Since u is not well defined during the integration, the best choice for the outcomes is $\dot{\alpha}_3/(\dot{\alpha}_1 + \dot{\alpha}_2 + \dot{\alpha}_3) > \hat{p}_3(u_{out})$. In the limit $\Omega \rightarrow -\infty$ these outcomes correspond to those defined for the Farey map. The full dynamics is, however, considered for a continuous time and the outcomes are defined not only on a surface of section. In Fig. 3 we show the basins of initial conditions for $u_{out} = 7$. Fig. 3 may be compared with Fig. 1. Small differences between these figures were expected, since the latter is an asymptotic approximation

³This implication is of typical validity and includes the extreme case, when the box-counting dimension of the repellor equals the phase space dimension. Pathological counterexamples can be constructed, but do not seem to represent realistic systems [12].

while the former is exact and refers to initial conditions taken near the maximum of expansion. The complicated structure of Fig. 3 is also present in any magnification of the basin boundary - see Fig. 4, where we show a 100 times magnification of Fig. 3. The numerical computation of the box-counting dimension corresponding to $u_{out} = 7$ results $D_0 = 1.87 \pm 0.01$, consistent with the approximated value. In fact, the differences between the approximated and full fractal dimensions are quite small, as can be seen in Fig. 2. Moreover, the system has the so called Wada property [13]: Every point on the boundary of a basin is also on the boundary of the other two basins. The corresponding fractal figures of CL do not present this property [6].

From the above analysis follows that the mixmaster dynamics is in fact chaotic. This conclusion is independent of the particular space-time parameters used because D_0 is invariant under space-time diffeomorphisms. The numerical values obtained for D_0 depend on the choices made for the outcomes. But, once found a set of outcomes for which the resulting D_0 is fractional and goes to the phase space dimension in the limit “outcomes $\rightarrow 0$ ”, the system is unequivocally chaotic.

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Figure 1: Portrait of the basins of the approximated dynamics for $u_{out} = 7$. The initial conditions were chosen on a grid of 400×400 , for $\alpha_1 = \alpha^{(a)}$, $\alpha_2 = \alpha^{(b)}$ and $\alpha_3 = \alpha^{(c)}$. Regions in gray, black and white correspond to orbits that escape for $\dot{\alpha}_3/(\dot{\alpha}_1 + \dot{\alpha}_2 + \dot{\alpha}_3) > 56/57$ with $\alpha_3 = \alpha^{(a)}$, $\alpha_3 = \alpha^{(b)}$ and $\alpha_3 = \alpha^{(c)}$, respectively.

Figure 2: Box-counting dimension (D_0) of the basin boundary (future invariant set) as a function of the outcome parameter (u_{out}) for: (a) Approximated dynamics (triangles); (b) Full dynamics (squares). The size of the triangles and squares corresponds approximately to the statistical uncertainty of D_0 .

Figure 3: Portrait of the basins of the full dynamics for $u_{out} = 7$. The initial conditions were chosen on a grid of 400×400 , for $\alpha_1 = \alpha^{(a)}$, $\alpha_2 = \alpha^{(b)}$ and $\alpha_3 = \alpha^{(c)}$. Regions in gray, black and white correspond to orbits that escape for $\dot{\alpha}_3/(\dot{\alpha}_1 + \dot{\alpha}_2 + \dot{\alpha}_3) > 56/57$ with $\alpha_3 = \alpha^{(a)}$, $\alpha_3 = \alpha^{(b)}$ and $\alpha_3 = \alpha^{(c)}$, respectively.

Figure 4: A 100 times magnification of a portion of Fig. 3.







